

Physics of Racing, Part 24:

Combination Slip

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The goal in this and the next installment of the *Physics of Racing* is to combine the magic formulae of parts 21 and 22, so that we have a model of tire forces when turning and braking or turning and accelerating *at the same time*. In this part, we figure out combination slip, and in the next installment, we figure out combination grip. Roughly speaking, slip is the input and grip is the output to our model. Slip comes from control inputs on brakes, throttle and wheel, grip comes from reaction forces of the ground on the tires.

The regular magic formulae apply only to a tire generating longitudinal or lateral forces in isolation, that is, to a tire accelerating or braking and *not* turning, or a tire turning but *not* accelerating or braking. In part 7, we approximated the response under combination slip by noting that it follows the circle of traction. A tire cannot deliver maximal longitudinal grip when it's delivering lateral grip at the same time, and *vice versa*. According to my sources, modeling of combination slip and grip is an area of active research, which means we are on our own, once again, in the original, risk-taking spirit of the *Physics of Racing* series. In other words, we're going way out on a limb and this could all be totally wrong, but I promise you lots of fun physics on the journey.

From part 21, recall our definition for the longitudinal slip, σ , the input to the longitudinal magic formula

$$\sigma = \frac{\omega}{\omega_0} - 1 = \frac{\omega}{V/R_e} - 1 = \frac{\omega R_e - V}{V}$$

where V is the forward speed of the hub w.r.t. ground, R_e is a constant, the **effective radius**, for a given tire, and ωR_e is the *backward* speed of the CP w.r.t. the hub. Therefore, $\omega R_e - V$ is the backward speed of the CP w.r.t. EARTH. A slick technique for proving this, and, in fact, for figuring out *any* combinations of relative velocities (see part 19) is as follows. Write $V = \text{HUB} - \text{GD}$, meaning speed of the HUB relative to the ground (GD). Now write $\omega R_e = -(\text{CP} - \text{HUB})$, meaning the backward speed of the CP relative to the HUB; the overall minus sign outside reminds us that we want ωR_e positive when the CP moves *backwards* w.r.t. the hub. Now, just do arithmetic:

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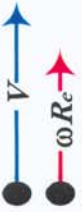
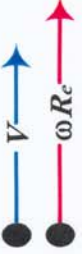




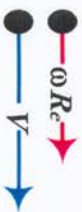

$$\begin{aligned}\omega R_e - V &= -(\text{CP} - \text{HUB}) - (\text{HUB} - \text{GD}) \\ &= -\text{CP} + \text{HUB} - \text{HUB} + \text{GD} = -(\text{CP} - \text{GD})\end{aligned}$$

voila, backward speed of the CP w.r.t. the ground. This realization gives us intuition into the sign of σ : if and only if the CP moves backwards faster than the hub moves forward; the car accelerates forward—visualize that in your head; in that case, ωR_e is greater than V ; $\omega R_e - V$ is greater than zero; and σ is positive.

It turns out that we developed this formula only for the case when V is positive, that is, the car is moving forward. And, in fact, the formula only works in that case. To generalize it to cars moving in reverse, we'd best analyze it in excruciating detail. A moment's reflection reveals that there are eight cases: two signs for V , two signs for ωR_e , and two cases for whether the absolute value of V is greater than the absolute value of ωR_e , yielding eight = $(2 \times 2 \times 2)$ possibilities, which have the following physical interpretations:

1. Car (hub) moving forward, CP moving slowly forward w.r.t. ground, resisting car motion. This models driving slowly in reverse gear while moving forward. **Slowly**, here, means slowly relative to V , or, precisely, that $|\omega R_e| < |V|$, where $|\omega R_e|$ is the absolute value or unsigned magnitude of ωR_e and $|V|$ is the absolute value or unsigned magnitude of V .
2. Same as above, just with CP moving **quickly** forward, that is with $|\omega R_e| > |V|$.
3. Car moving forward, CP moving slowly backwards, just not quickly enough to accelerate the car. This is braking or engine braking in forward gear. Wheel lockup while moving forward falls in this case, too.
4. Car moving forward, CP moving quickly backwards, accelerating the car forward.
5. Car moving backward, CP moving slowly forwards, just not quickly enough to accelerate the car backward. This is braking or engine braking in reverse, and wheel lockup in reverse falls in this case.
6. Car moving backward, CP moving quickly forward, accelerating in reverse.
7. Car moving backward, CP moving slowly backward, resisting motion.
8. Car moving backward, CP moving quickly backward, resisting motion.

We've caught all this in the following diagram, in which we have drawn V and ωR_e as arrows, pointing in the actual direction that the hub moves w.r.t. the ground and the CP moves w.r.t. the hub, respectively. Algebraically, V and ωR_e have opposing sign conventions, so ωR_e is negative when its arrow points up. In looking at this table, note that the longitudinal force F_x has the same sign as σ . When F_x and σ are positive, the car is being forced forward by the ground's reacting to the tires. When they're negative, the car is being forced backwards. So, to figure out which way the car is being forced, just look at the sign of σ .

case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
<i>Spinning Slowly Forward</i>	<i>Spinning Fast Forward</i>	<i>Spinning Slowly Backward</i>	<i>Spinning Fast Backward</i>	<i>Spinning Slowly Forward</i>	<i>Spinning Fast Forward</i>	<i>Spinning Slowly Backward</i>	<i>Spinning Fast Backward</i>
<i>Moving Forward</i>				<i>Moving Backward</i>			
$V > 0$	$V > 0$	$V > 0$	$V > 0$	$V < 0$	$V < 0$	$V < 0$	$V < 0$
$\omega R_e < 0$	$\omega R_e < 0$	$\omega R_e > 0$	$\omega R_e > 0$	$\omega R_e < 0$	$\omega R_e < 0$	$\omega R_e > 0$	$\omega R_e > 0$
$\omega R_e - V < 0$	$\omega R_e - V < 0$	$\omega R_e - V < 0$	$\omega R_e - V > 0$	$\omega R_e - V > 0$	$\omega R_e - V < 0$	$\omega R_e - V > 0$	$\omega R_e - V > 0$
$\sigma < 0$	$\sigma < 0$	$\sigma < 0$	$\sigma > 0$	$\sigma > 0$	$\sigma < 0$	$\sigma > 0$	$\sigma > 0$
							

Inspection of this table reveals that the following *new* formula works in all cases:

$$\sigma = \frac{\omega R_e - V}{|V|}$$

Where the numerator, $\omega R_e - V$ is the *signed* difference of the two speeds and the denominator is *unsigned*. It is perhaps surprising that there is so much richness in such a little formula. However, it is precisely this richness that we must maintain as we add steering, that is, lateral slip angle at the same time. The best way to do that is to vectorize the formula so that the algebraic signs of the vector components take the place of the signed quantities V and ωR_e . The approach here parallels the approaches of parts 16 and 19. We want the signed component V_x to take the place of the old, signed V , the signed component L_x of the slip velocity \mathbf{L} to take the place of the old ωR_e , and V *now* to denote the unsigned magnitude of the vector \mathbf{V} , that is $V \equiv \|\mathbf{V}\| \equiv \sqrt{V_x^2 + V_y^2 + V_z^2}$. The next table summarizes these changes:

<i>Quantity</i>	<i>old notation</i>	<i>new notation</i>	<i>vector</i>
signed, forward speed of hub w.r.t. EARTH	V	V_x	\mathbf{V}
signed, backward speed of CP w.r.t. hub	ωR_e	$-W_x$	\mathbf{W}
unsigned magnitude of hub speed	$ V $	V	$\ \mathbf{V}\ $ or V
signed, backward speed of CP w.r.t. EARTH	$\omega R_e - V$	$-L_x$	$\mathbf{L} = \mathbf{V} + \mathbf{W}$
signed, longitudinal slip	$\sigma = \frac{\omega R_e - V}{ V }$	$\sigma = \frac{-L_x}{V}$?

Slip velocity, $\mathbf{L} \equiv [L_x, L_y, L_z] = \mathbf{V} + \mathbf{W}$ is the plain-old vector velocity of the CP w.r.t. EARTH with no secret sign convention to confuse things. As an aside, we note that when the car sticks to the ground on flat road, we may assume $L_z = 0$. \mathbf{W} is CP velocity w.r.t. hub. In the TIRE system, \mathbf{W} has only a (signed) x-component, that is, $\mathbf{W}_{\text{TIRE}} = [W_x, 0, 0]$. These definitions hold whether the car is moving forward or backward, accelerating or braking.

The big question mark in the table indicates that we do not have a vector for combination slip because we measure its longitudinal and lateral components differently, as a ratio and as an angle, respectively. Note that, since lateral slip α is the angle made by \mathbf{V} in the TIRE system, it is $\tan^{-1}(V_{y, [\text{TIRE}]} / V_{x, [\text{TIRE}]})$. Since $\mathbf{L} = \mathbf{V} + \mathbf{W}$, it's easy to see that

$\alpha = \tan^{-1}\left(\frac{L_y}{L_x - W_x}\right)$, which is a most convenient expression, though some attention must be paid to the quadrant in which the angle falls. We resolve this in the next two installments of *PbORS* as we stitch together the two magic formulae to make *Combination Grip*.

But first, let's update the big diagram, showing all eight cases with a little slip angle thrown into the mix, and the vector sum, $\mathbf{L} = \mathbf{V} + \mathbf{W}$, replacing the *ad hoc*, signed quantities of the old notation. The sign of the slip angle α does not introduce new cases so long as $|\alpha| < 90^\circ$ because the right-hand and left-hand cases are precisely symmetrical. The nice thing, here, is that we can treat all eight cases the same way—the nature of vector math takes care of it because the magnitude of a vector is always unsigned. Using signed, scalar quantities, we had to dissect the system and introduce *absolute value* to get everything to work. Absolute value has always struck me as a kind of crock or kludge to use when the math is just not sufficiently expressive. The main contribution of this installment is to fix that problem.

case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
<i>Spinning Slowly Forward</i>	<i>Spinning Fast Forward</i>	<i>Spinning Slowly Backward</i>	<i>Spinning Fast Backward</i>	<i>Spinning Slowly Forward</i>	<i>Spinning Fast Forward</i>	<i>Spinning Slowly Backward</i>	<i>Spinning Fast Backward</i>
Moving Forward				Moving Backward			
$V_x > 0$	$V_x > 0$	$V_x > 0$	$V_x > 0$	$V_x < 0$	$V_x < 0$	$V_x < 0$	$V_x < 0$
$-W_x < 0$	$-W_x < 0$	$-W_x > 0$	$-W_x > 0$	$-W_x < 0$	$-W_x < 0$	$-W_x > 0$	$-W_x > 0$
$-L_x < 0$	$-L_x < 0$	$-L_x < 0$	$-L_x > 0$	$-L_x > 0$	$-L_x < 0$	$-L_x > 0$	$-L_x > 0$
$\sigma < 0$	$\sigma < 0$	$\sigma < 0$	$\sigma > 0$	$\sigma > 0$	$\sigma < 0$	$\sigma > 0$	$\sigma > 0$